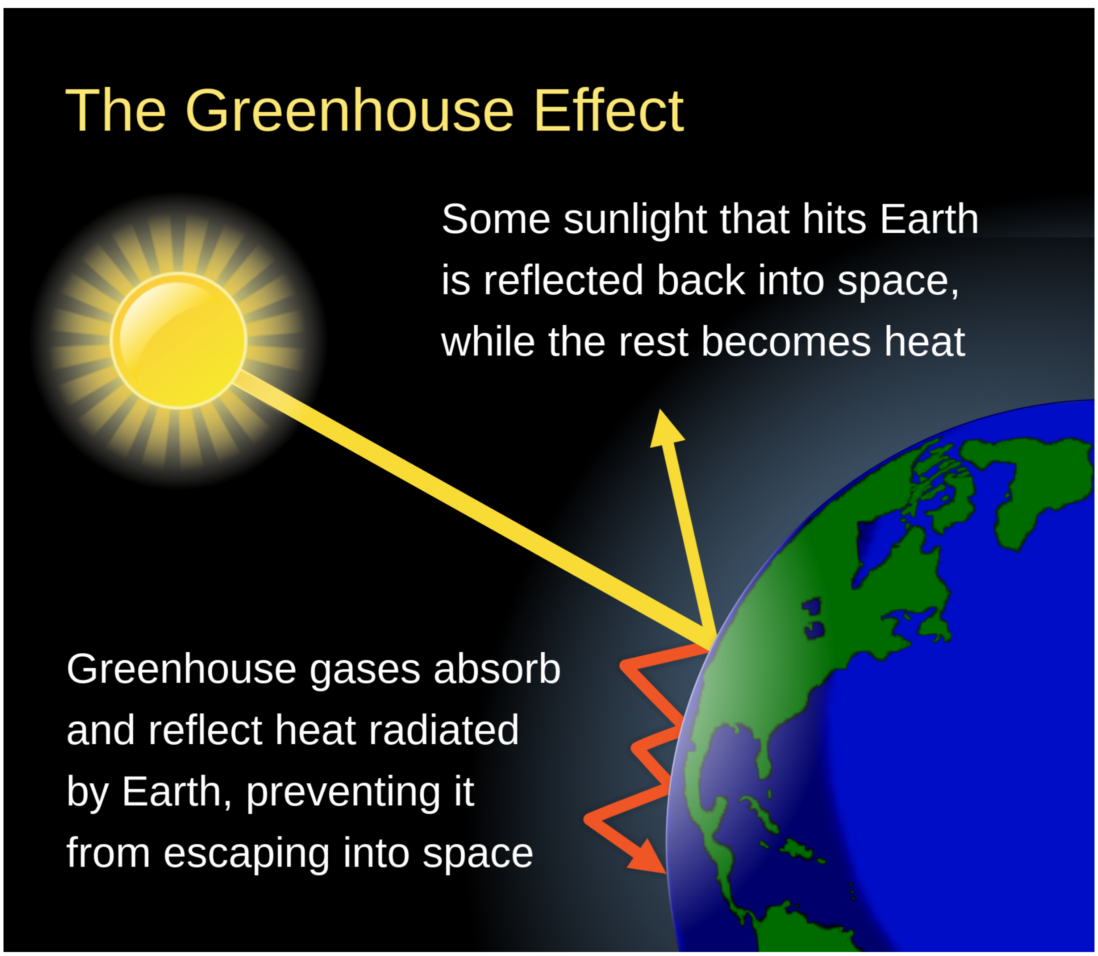
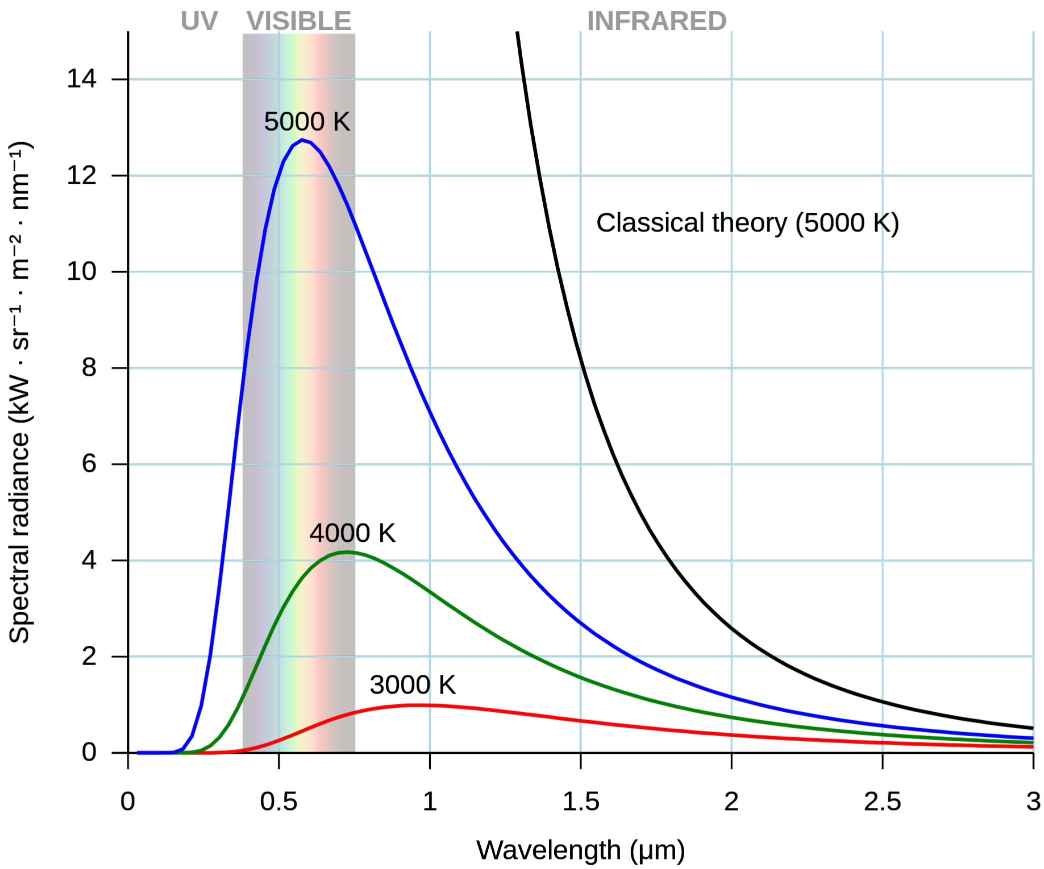
Grade 11-12 Math Circles
Nov. 16, 2022
The Mathematics of Climate Change 3

Introductory Comments

Last week we discussed some basics of the black body idea, and how the equilibrium temperature of a black body Earth is found. We concluded, somewhat sadly, that a black body Earth is below freezing. This week we will discuss how the addition of an atmosphere allows us to construct a black body model that gets the temperature "right". The key idea, which we ignored last time, is the notion that energy is radiated at different wavelengths (e.g. the different colours in the visible part of the electromagnetic spectrum) and some wavelengths are captured by the atmosphere rather than lost to space.



This image shows the schematic of this so–called **Greenhouse Effect**. This is a stupidly human–centered term since the atmosphere was doing the job for millions of years before anyone every built a greenhouse! We start with Planck's law for radiation by wavelength. The figure below shows "hot" objects like the Sun, for which the peak energy occurs in the visible spectrum. You can see that as the temperature drops the peak decreases (Stefan's law tells you that) and shifts to longer wavelengths (Stefan's does not tell you that). It is this shift that is the key. It means that if the atmosphere catches infrared energy but not the visible part of the electromagnetic spectrum, then radiation coming in could heat the surface but the radiated energy from the ground may not all make it out to space.



Stop and Think

Can you think of other physical systems which have waves of different lengths? Can you think of "filters" that catch only some of the length scales?

The Simplest Greenhouse Earth

We need to revisit our energy balance model, this time for two different temperatures in three different places: one at the top of the atmosphere, one that represents the average state of the atmosphere, T_a, and one that represents the average state of the ground, T_s. We then write energy balances for each of these. For the top of the atmosphere we have something that is a slightly modified version of what we saw last week (I also make a nod to astrophysical convention and what I called S last week I relabel as $S/4$. I am happy to discuss why on the board),

$$\begin{aligned}\text{Energy from Sun absorbed} - \text{ Energy radiated out from atmosphere} + \\ + \text{ Energy radiated out from ground that is NOT absorbed by the atmosphere} = 0\end{aligned}$$

and now as an equation

$$\frac{S}{4}(1-\alpha) - \sigma T^4 - (1-\epsilon)\sigma T_s^4 = 0.$$

Here we have a new parameter, ϵ, which gives us the fraction of radiation in the infrared band that is absorbed by the atmosphere. For the atmosphere we have in words

$$\begin{aligned}-\text{Energy from atmosphere radiated up} - \text{ Energy from atmosphere radiated down} + \\ + \text{ Energy radiated out from ground that is absorbed by the atmosphere} = 0\end{aligned}$$

or in equation form

$$2\epsilon\sigma T_a^4 = \epsilon\sigma T_s^4$$

and we see right away that we can solve this to get

$$T_a = 2^{-1/4} T_s$$

which is very convenient indeed!

OK finally we need to consider the ground. Again, starting in words we have

$$\begin{aligned}\text{Energy from space that passes through the atmosphere} + \text{ Energy from atmosphere radiated down} - \\ - \text{ Energy radiated out from ground} = 0\end{aligned}$$

and in equation form

$$\frac{S}{4}(1-\alpha) + \epsilon\sigma T_a^4 - \sigma T_s^4 = 0$$

Solving the Model

I put a section break here again to remind the reader that the process of modelling and the process of solving the model are often completely separate exercises. I record the answer here, but the solution of the modelling above is the subject of Exercise 1. Here I just record the solution.

$$T_s = 2^{-1/4}\left(\frac{S(1-a)}{\sigma(2-\epsilon)}\right)^{1/4}$$

Activity 1

Try some parameters to get numerical answers for the problem. First recall that $S = 1370$, $a = 0.3$ and $\sigma = 5.6704 \times 10^{-8}$, then try $\epsilon = 0.77$, for a realistic estimate of what the atmosphere does. Then go back and recover the black body result (discuss with some neighbours on what you should set ϵ to be for this case), finally try a super thick atmosphere (like Venus might have) $\epsilon = 0.99$.

Discussion: (you can record what we talk about here):

Activity 2

This exercise is a bit about "real world" math problems. I want you to discuss, then write out what made it possible to solve the model as a formula. If the model only gave you two coupled quartics (fourth order polynomials) would you be able to solve them? Why or why not?

Discussion: (you can record what we talk about here):

Exercise 1

Do the algebra to solve the model for the ground temperature.

Exercise 1 Solution

Here I just record the steps. First note that

$$T_a = 2^{-1/4} T_s$$

and then substitute this into either of the equations for the ground or top of atmosphere and simplify.

Exercise 2

How does the real atmosphere differ from the simple box model? Try to list differences in order of importance.

Exercise 2 Solution

The first and foremost difference is that the atmosphere can move.
The second difference is that the amount of energy coming in varies with latitude and by season.
It is the job of the first difference to "even out" some of the differences; heat is transported northward making even regions that get far less solar insolation (like K-W) habitable.
Finally, all the physics we have talked about ignored water vapour, which can change from liquid to gas and back again. Water vapour, in the form of clouds lets us track atmospheric motions and is one of the big challenges as far as modelling climate and weather is concerned.